differential equation

ordinary

$y(x)$

$y'(x)$, $y''(x)$

partial

more than one independent variable

Order → higest order derivative

degree → power of height derivative

Gen. for = $f(x, y, y', y'' \cdots) = Q$

linear diff eq → degree 1. i.e. power of $y = 1$.

$$P_n(x)\frac{d^n(y)}{dx^n} + \cdots\cdots + y(x) = Q$$

if $Q = 0$, then it is called homogenous equation.

Types of solution: →

1. Gen. sol" → contains arbitary const.
2. particular sol" → initial cond" $y(0) = 1$ → do not contain ~~const~~ cont.
3. singular sol" → does no contains arb. const and can not be derived from gen. sol.

Equation of first order and first degree equation: →

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad M\,dx + N\,dy = 0$$

a). variable separable method: →

$$\frac{dy}{dx} = f(x,y) = g(x).h(y) \Rightarrow \int \frac{dy}{h(y)} = \int g(x)\,dx$$

b). Reducable to variable sep. →

$$\frac{dy}{dx} = f(ax + by + c)$$ assume

$ax+by+c = z$

$a + by' = z'$

$$\boxed{y' = \frac{1}{b}(z'-a)}$$

$$\boxed{\frac{1}{b}(z'-a) = f(z)}$$

$\cancel{\frac{z'}{b} + \frac{a}{b} = f(z)}$

$\cancel{\frac{z'}{b\,f(z)} = a}$

c). Homogenous equation: →

degree same

$$f(kx, ky) = k f(x,y).$$

Ex.

$$f(x,y) = x^4 + x^2y^2 + y^3x + x^3y$$

$$f(kx, ky) = (kx)^4 + k^4(x^2y^2) + k^4 y^3 x + k^4 x^3 y$$

$$\boxed{f(kx, ky) = k^4 f(x,y).}$$

$$\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$$ ⇒ f & ϕ are fun" of x & y having same deg".

to solve these que. put $\boxed{y = vx}$

d). Equation reducible to homogenous eqⁿ :→

let
$$\frac{dy}{dx} = \frac{ax_1 + by_1 + c_1}{ax_2 + by_2 + c_2}$$

to make these type eqⁿ to homo. eqⁿ

replace $y = y + K$, & $x = x + h$

Ex:
$$\frac{dy}{dx} = \frac{x+y+2}{x-y-3}, \quad \text{put } x = X + h, \; y = Y + K$$

$$\frac{dy}{dx} = \frac{X + Y + h + K + 2}{X - Y + h - K - 3}$$

Choose h, k such that
$$\begin{aligned} h + K + 2 &= 0 \\ h - K - 3 &= 0 \end{aligned} \qquad h = 1/2, \quad K = -5/2$$

then we get
$$\frac{dy}{dx} = \frac{X+Y}{X-Y} \rightarrow \text{now put } \boxed{Y = Vx}$$

and at last again put $V = \frac{Y}{x}$ and $Y = Y - 5/2$
$X = X + 1/2$

e). linear differential equation:→

$$\frac{dy}{dx} + Py = Q$$

NOTE:→
P & Q are the function of x only.

step. 1. $I.f = e^{\int P dx}$

step. 2 $y(I.f) = \int Q(If)\, dx + c$

f). Equation reducible to linear differential eqⁿ :→

- $\frac{dy}{dx} + Py = Qy^n$ to reduce it divide both side with y^{1-n}

put $\boxed{y^{1-n} = z}$, then you will find you eq^n in linear form.

9). Exact differential equation:→

$$Mdx + Ndy = 0$$

a eq^n will be exact iff $\frac{dM}{dy} = \frac{dN}{dy}$

Sol^n

$$\int M dx + \int N dy = c$$

treat y as const — term not containing x

9.1). Reducible to exact differential equation:→ $(M_y \neq N_y)$

Rule 1:→ if $Mdx + Ndy = 0$ (homogeneous eq^n i.e total Power of x & y same)

$$I.f = \frac{1}{MX+NY}$$

Rule.2:→ if the given eq^n is type of $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

then $I.f. = \frac{1}{MX-NY}$

Ex: $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0$

Rule.3

if $\frac{M_y - N_x}{N}$ → is a fun^n of x only

then.

$$I.f = e^{\int \frac{M_y - N_x}{N} \cdot dx}$$, Here $M_y = \frac{\partial M}{\partial y}$ & $N_x = \frac{\partial N}{\partial x}$

Rule.4.

if $\frac{N_x - M_y}{M} = f(y)$ is a fun^n of y only

then.

$$I.f = e^{\int f(y) \cdot dy}$$

Equation of 1st order but higher degree :→

$$P_0 \left(\frac{dy}{dx}\right)^n + P_1 \left(\frac{dy}{dx}\right)^{n-1} + \cdots \cdots + P_n = 0$$

a). ~~rule 1~~: types: →

(i) equation solvable for p, here $p = \frac{dy}{dx}$

(ii) " solvable for y

(iii) " solvable for x

(iv) clairaut's eq^n $y = px + f(p)$

or $y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$

Sol^n (iv) replace $p = c$

$$= \boxed{y = cx + f(c)}$$

Reducible to claiout eq": →

Ex: $(xy' - y)(y'y + x) = a^2 y'$

put $x^2 = v$, $y^2 = u$

$2x\,dx = dv$, $2y\,dy = du$

$\frac{dy}{dx} = \frac{x}{y}\frac{du}{dv}$,

NOTE; This Method work only for this question every time we have to think.

linearly independent / linearly dependent solution: →

linearly combination: →

$$y'' + Py' + Qy = R \quad (1)$$

let y_1 , y_2 — y_1 & y_2 sol" of eq" (1)

then. $y = c_1 y_1 + c_2 y_2$ will be another sol".

if $y \neq c y_2$ [linearly independent

or $c_1 y_1 + c_2 y_2 = 0$ iff $c_1 = c_2 = c_2$]

if $c_1 y_1 + c_2 y_2 = 0$ possible for c_1 & c_2 not necessarily 0.

if $f_1, f_2 \ldots f_n$ are L.D.
then any of them can be written as L.C. of other.

Wronskian: →

$$W = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{n-1} \end{vmatrix}$$

if $W \neq 0$ then $f_1, f_2 \ldots f_n$ are linearly independent over the interval.
& if $W = 0$ then $f_1, f_2 \ldots f_n$ are linearly dep.

Ex. $\sin^2 x, \cos^2 x, 1.$ → L.D.
As. $\sin^2 x + \cos^2 x = 1.$

Higher order differential equation: →

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = P_3$$ P_1, P_2 & P_3 are the fun" of x

⇓

y_1 & y_2 are sol" of the eq"

then $y = c_1 y_1 + c_2 y_2$ are L.I. is a sol"

for nth order

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

Gen. solⁿ

$y = CF + PI$

Complimentary funⁿ ; particular solⁿ

Auxillary eqⁿ

$$\left\{ \frac{d}{dx} = D, \quad \frac{d^2}{dx^2} = D^2 \right\}$$

linear differential equation with const coefficient:→

i, n^{th} order 1st degree:→

Roots of Auxillary eqⁿ	Corresponding parts of complimentary solution.
1 one real root m_1	$c_1 e^{m_1 x}$
2. two distint real root m_1 & m_2	$c_1 e^{m_1 x} + c_2 e^{m_2 x}$
3). two same real roots m_1, m_2	$(c_1 + c_2 x) e^{m_1 x}$
4) three same real roots m_1, m_1, m_1	$(c_1 + c_2 x + c_3 x^2) e^{m_1 x}$
5). Imaginary roots $\alpha \pm i\beta$	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$
6). two pair of Imaginary but same roots $\alpha \pm i\beta$, $\alpha \pm i\beta$	$e^{\alpha x}\left[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x\right]$

particular integral

$f(D)y = Q$

$y = \frac{Q}{f(D)} \rightarrow PI$

&

$\frac{1}{D} Q = \int Q\,dx$

with no arb. const

$\frac{1}{D-a} Q = e^{ax} \int Q e^{-ax} dx$

five standard method of particular sol^n :→

1. $\frac{1}{f(D)} e^{ax}$ put $D = a$

case of failure of $f(a)$

then

$x \frac{1}{\frac{d}{dD}(f(D))} e^{ax}$

if again failure. the $x^2 \frac{1}{\frac{d^2}{dD^2}(f(D))} e^{ax}$

2. $\frac{1}{f(D^2)}$ sin ax or cos ax then put $D^2 = -a^2$

3). $\frac{1}{f(D)} x^m$ use binomial th^m

$(1-D)^{-1} = 1 + D + D^2 + \cdots$

$(1+D)^{-1} = 1 - D + D^2 - \cdots$

$(1-D)^{-2} = 1 + 2D + \cdots$

4). $\frac{1}{f(D)} e^{ax} X$ $\Rightarrow$ eq^n $e^{ax} \frac{1}{f(D+a)} X$ — X is a fun^n of x

5). $\frac{1}{f(D)} (xX)$ $\Rightarrow$ sol^n $x \frac{1}{f(D)} X + \frac{d}{dD}\left(\frac{1}{f(D)} X\right)$

linear diff. eq^n

Method of undetermind coefficient superposition approach

given non. homogenous eq $a_n y^n + a_{n-1} y^{n-1} + \cdots + a_0 y = g(x)$ — (1)

to solve the eq^n

find a) $y_c \rightarrow$ complimentry fun.

b) $y_p \rightarrow$ particular sol^n.

Ex: $\rightarrow$

Linear Diff.

To solve a non homogenous equation $a_n y^n + a_{n-1} y^{n-1} + \dots a_0 y = g(x)$ → (1)

(i) find Complementry fun y_c

(ii) . find particular sol^n y_p of (1)

so that $y = y_c + y_p$ is gen. sol^n of (1)

method of undetermined coeff. superposition approach.

we develop a mothod to find y_p . motivated by the kinds of fun^n that make up input fun. $g(x)$.

ideas is to have an educated guess(not with a wild guess) when $g(x)$ is of certain types.

Ex. solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$

Sol^n $y = y_c + y_p$

y_c is the sol^n of $y'' + 4y' - 2y = 0$

assume sol^n $y = e^{mx}$

Step-1

Auxillary eqution $m^2 + 4m - 2 = 0$

$m = -2 \pm \sqrt{6}$

$y_c = c e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x}$

step. 2. find y_p

$\because$ $g(x)$ is a quadratic polynomial

assume $y_p = Ax^2 + Bx + C$

we see A, B, C s.t. y_p is a sol^n of given ODE.

$y_p' = 2Ax + B$ & $y_p'' = 2A$

$\Rightarrow$ $2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$

$= -2Ax^2 + (8A - 2B)x + 2A + 4B - 2C = 2x^2 - 3x + 6$

on Comp.

$-2A = 2$

$\boxed{A = -1}$

$8A - 2B = -3$

$-8 - 2B = -3$

$2B = -5$

$\boxed{B = -5/2}$

$2(-1) + 4\left(-\frac{5}{2}\right) - 2C = 6$

$-2 - 10 - 2C = 6$

$\frac{-12 - 6}{2} = C$

$\boxed{C = -9}$

$y_p = -x^2 - \frac{5}{2}x - 9 = 0$

$y_p = 2x^2 + 5x + 18 = 0$

Hint

$L(y) = g_1(x) + g_2(x)$

$y_1 = L(y) = g_1(x)$

$y_2 = L(y) = g_2(x)$

Denis Zhell cullen book

Trail particular sol^n

$g(x)$	forms of y_p
k (constant)	A
$5x+7$	$Ax+B$
$3x^2-2$	Ax^2+Bx+C
x^3-x+1	Ax^3+Bx^2+Cx+d
$\sin 4x, \cos 4x$ ⇓ $\sin ax, \cos ax$	$A\cos ax + B\sin ax$
e^{mx}	Ae^{mx} or Axe^{mx}
$(ax+b)e^{mx}$	$(Ax+B)e^{mx}$
$ax^2 e^{mx}$	$(Ax^2+Bx+C)e^{mx}$
$e^{ax}\sin bx$	$Ae^{ax}\cos bx + Be^{ax}\sin bx$
$ax^2\sin bx$	$(Ax^2+Bx+C)\cos bx + (Ex^2+Fx+G)\sin bx$

NOTE :→ this method is not applicable to fun^n $g(x)$

$g(x) = \ln(x)$, $\frac{1}{x}$, $\tan x$, $\tan^{-1}x$

Ex find particular sol^n $y''-5y'+4y = 8e^x$

y_c: $y''-5y'+4y=0$

$m^2-5m+4=0$, $m=1,4$

$$y_c = c_1 e^{x} + c_2 e^{4x}$$

particular sol^n

$$y_p(x) = Axe^{x}$$

$$y_p'(x) = Axe^{x} + Ae^{x} = Ae^{x}(x+1)$$

$$y_p'' = Ae^{x}(x+1) + Ae^{x}$$

$$y_p'' = Ae^{x}[x+2]$$

$$A\cancel{e^{x}}[x+2] - 5[A\cancel{e^{x}}(x+1)] + 4Ax\cancel{e^{x}} = 8\cancel{e^{x}}$$

$$\underline{Ax} + 2A - \underline{5Ax} - 5 + \underline{4Ax} = 8$$

$$2A - 5 = 8$$

$$\boxed{A = \frac{13}{2}} \text{ Â rechecks}$$

Remarks: → The form y_p is a set of all linearly independent function that are generated by repeated differentiations of $g(x)$.

Method. 2 variation of parameters: →

Summary: → we examine a method for determining $y_p(x)$ of non homogeneous DE, that has, in theory no restrictions on it

Ex solve $a_2 y'' + a_1 y' + a_0 y = g(x)$ By Cramers rules

Step. 1 put into standard form $y'' + Py' + Qy = f(x)$.

Step. 2. they then you find y_c of eq^n in step. 1

$$y_c = c_1 y_1 + c_2 y_2$$

Step. 3 Calculate $W[y_1(x), y_2(x)]$,

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Step 4

Calculate u_1 & u_2 by solving

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$

Step. 5

$$y_p = u_1 y_1 + u_2 y_2$$

Step. 6 $\boxed{y = y_c + y_p}$

let L is an operator such that

$$L \equiv a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0$$

$$L(y) = f(x).$$

Annehilator approach to solve non-homogenous ODE →

Ex:

$$\underbrace{y'' + 3y' + 2y}_{f(x)} = \underbrace{4x^2}_{g(x)} \quad \text{— (A)}$$

step.1: solve corresponding homo. eq. $f(x) = y''+3y'+2y = 0$

Auxulary eq: $M^2 + 3M + 2 = 0 \Rightarrow M = -2, -1,$

$$y_c = c_{11}e^{-2x} + c_{21}e^{-x}$$

step.2: → find an operator L such that $L\{g(x)\} = 0$

i.e $L\{4x^2\} = 0 \Rightarrow L \equiv D^3$

Apply L on both side:

$$D^3(y'' + 3y' + 2y) = 0 \quad \text{— (B)}$$

$$D^3(D^2 + 3D + 2)y = 0$$

Auxullary eq": → $M^3(M^2 + 3M + 2) = 0$

$$\Rightarrow M = 0, 0, 0, -2, -1.$$

$$y = y_c + y_p$$

$$\underbrace{c_{11}e^{-2x} + c_{21}e^{-x}}_{y_c} + \underbrace{(c_1 + c_2x + c_3x^2)}_{y_p}$$

$y_p = c_1 + c_2x + c_3x^2$

Substitute y_p, y_p', y_p'' into (A)

then compare you will get value of c_1, c_2 & c_3.

then again put value into $\boxed{y = y_p + y_c}$

then this y is the gen. sol^n of eq (A)